Problem 2.1

(a) Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face). What is the net force on a test charge Q at the center?

(b) Suppose *one* of the 12 qs is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.

(c) Now 13 equal charges, q, are placed at the corners of a regular 13-sided polygon. What is the force on a test charge Q at the center?

(d) If one of the 13 qs is removed, what is the force on Q? Explain your reasoning.

Solution

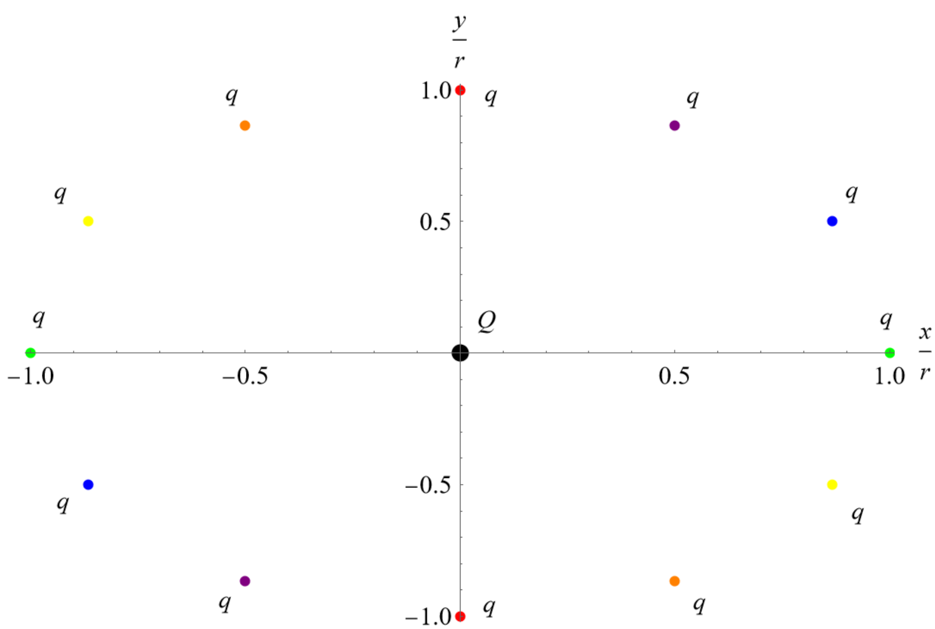
It's far more useful to think of the charges as being equally spaced on a circle with radius r rather than at the corners of a regular polygon because then it's obvious the charge positions can be described with polar coordinates.

Part (a)

Suppose there are 12 charges equally spaced on a circle with radius r, one of them being at the 6 o'clock position. The position vector of charge q_k is

$$\mathbf{r}'_k = r\left\langle \cos\left(-\frac{\pi}{2}+\frac{2\pi k}{12}\right), \sin\left(-\frac{\pi}{2}+\frac{2\pi k}{12}\right)\right\rangle, \quad 0 \le k \le 11.$$

The charges are illustrated below.



Those of the same color exert equal and opposite forces at the origin and therefore cancel out. The net force at the center is zero because of the radial symmetry.

Confirm this by using the principle of superposition and adding the electric forces vectorially.

$$\begin{aligned}
\mathbf{F} &= \sum_{k=0}^{11} \frac{1}{4\pi\epsilon_0} \frac{q_k Q}{\mathscr{r}_k^2} \hat{\mathscr{r}}_k \\
&= \sum_{k=0}^{11} \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \left(\frac{\mathbf{0} - \mathbf{r}'_k}{r} \right) \\
&= \frac{qQ}{4\pi\epsilon_0 r^3} \sum_{k=0}^{11} (-\mathbf{r}'_k) \\
&= -\frac{qQ}{4\pi\epsilon_0 r^3} \sum_{k=0}^{11} \mathbf{r}'_k \\
&= -\frac{qQ}{4\pi\epsilon_0 r^3} \sum_{k=0}^{11} r \left\langle \cos\left(-\frac{\pi}{2} + \frac{2\pi k}{12}\right), \sin\left(-\frac{\pi}{2} + \frac{2\pi k}{12}\right) \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \sum_{k=0}^{11} \left\langle \cos\frac{(k-3)\pi}{6}, \sin\frac{(k-3)\pi}{6} \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \left\langle \sum_{k=0}^{11} \cos\frac{(k-3)\pi}{6}, \sum_{k=0}^{11} \sin\frac{(k-3)\pi}{6} \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \left\langle \cos\frac{(-3)\pi}{6} + \cos\frac{(-2)\pi}{6} + \cdots + \cos\frac{(8)\pi}{6}, \sin\frac{(-3)\pi}{6} + \sin\frac{(-2)\pi}{6} + \cdots + \sin\frac{(8)\pi}{6} \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \langle 0, 0 \rangle \\
&= \mathbf{0}
\end{aligned}$$

Part (b)

If the charge at 6 o'clock is removed, then the downward force from the 12 o'clock charge remains uncancelled: $\mathbf{F} = -qQ/(4\pi\epsilon_0 r^2)\hat{\mathbf{y}}$.

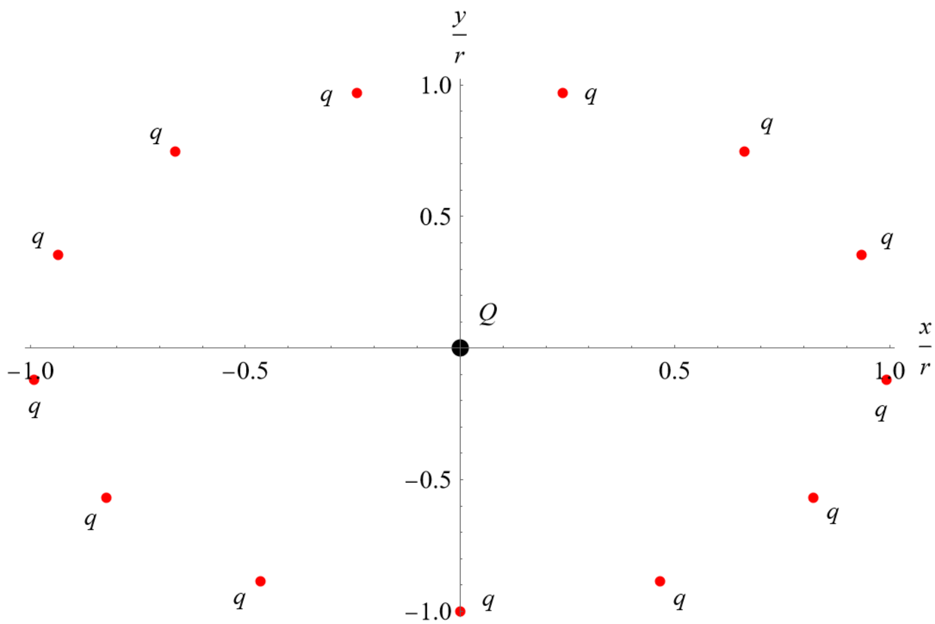
$$\begin{aligned}
\mathbf{F} = \sum_{k=1}^{11} \frac{1}{4\pi\epsilon_0} \frac{q_k Q}{\mathscr{r}_k^2} \hat{\mathscr{r}}_k &= -\frac{qQ}{4\pi\epsilon_0 r^2} \sum_{k=1}^{11} \left\langle \cos\frac{(k-3)\pi}{6}, \sin\frac{(k-3)\pi}{6} \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \left\langle \sum_{k=1}^{11} \cos\frac{(k-3)\pi}{6}, \sum_{k=1}^{11} \sin\frac{(k-3)\pi}{6} \right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \langle 0, 1 \rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2} \hat{\mathbf{y}}
\end{aligned}$$

Part (c)

Suppose that now there are 13 charges equally spaced on a circle with radius r, one of them being at the 6 o'clock position. The position vector of charge q_k is

$$\mathbf{r}'_k = r\left\langle \cos\left(-\frac{\pi}{2} + \frac{2\pi k}{13}\right), \sin\left(-\frac{\pi}{2} + \frac{2\pi k}{13}\right)\right\rangle, \quad 0 \le k \le 12.$$

The charges are illustrated below.



The force at the center is still zero because of the radial symmetry—the charges are equally spaced around the circle. Confirm this by using the principle of superposition and adding the electric forces vectorially.

$$\begin{aligned}
\mathbf{F} = \sum_{k=0}^{12} \frac{1}{4\pi\epsilon_0}\frac{q_k Q}{\mathscr{r}_k^2}\hat{\boldsymbol{\mathscr{r}}}_k &= \sum_{k=0}^{12} \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\left(\frac{\mathbf{0}-\mathbf{r}'_k}{r}\right) \\
&= \frac{qQ}{4\pi\epsilon_0 r^3}\sum_{k=0}^{12}(-\mathbf{r}'_k) \\
&= -\frac{qQ}{4\pi\epsilon_0 r^3}\sum_{k=0}^{12}\mathbf{r}'_k \\
&= -\frac{qQ}{4\pi\epsilon_0 r^3}\sum_{k=0}^{12} r\left\langle \cos\left(-\frac{\pi}{2} + \frac{2\pi k}{13}\right), \sin\left(-\frac{\pi}{2} + \frac{2\pi k}{13}\right)\right\rangle \\
&= -\frac{qQ}{4\pi\epsilon_0 r^2}\sum_{k=0}^{12}\left\langle \cos\frac{(4k-13)\pi}{26}, \sin\frac{(4k-13)\pi}{26}\right\rangle
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}\mathbf{F} &= -\frac{qQ}{4\pi\epsilon_0 r^2}\left\langle \sum_{k=0}^{12}\cos\frac{(4k-13)\pi}{26}, \sum_{k=0}^{12}\sin\frac{(4k-13)\pi}{26}\right\rangle \\ &= -\frac{qQ}{4\pi\epsilon_0 r^2}\langle 0, 0\rangle \\ &= \mathbf{0}\end{aligned}$$

Part (d)

If the charge at 6 o'clock is removed, then the remaining 12 charges conspire to produce the same force as before: $\mathbf{F} = -qQ/(4\pi\epsilon_0 r^2)\hat{\mathbf{y}}$.

$$\begin{aligned}\mathbf{F} = \sum_{k=1}^{12}\frac{1}{4\pi\epsilon_0}\frac{q_k Q}{\mathscr{r}_k^2}\hat{\mathscr{r}}_k &= -\frac{qQ}{4\pi\epsilon_0 r^3}\sum_{k=1}^{12} r\left\langle \cos\left(-\frac{\pi}{2}+\frac{2\pi k}{13}\right), \sin\left(-\frac{\pi}{2}+\frac{2\pi k}{13}\right)\right\rangle \\ &= -\frac{qQ}{4\pi\epsilon_0 r^2}\left\langle \sum_{k=1}^{12}\cos\frac{(4k-13)\pi}{26}, \sum_{k=1}^{12}\sin\frac{(4k-13)\pi}{26}\right\rangle \\ &= -\frac{qQ}{4\pi\epsilon_0 r^2}\langle 0, 1\rangle \\ &= -\frac{qQ}{4\pi\epsilon_0 r^2}\hat{\mathbf{y}}\end{aligned}$$